Xeno Semantics for Ascending and Descending Truth

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0. Introduction

As part of an approach to the liar paradox and the other paradoxes affecting truth, I have proposed replacing our concept of truth with two concepts: ascending truth and descending truth.[1] I am not going to discuss why I think this is the best approach or how it solves the paradoxes; instead, I concentrate on the theory of ascending and descending truth. I formulate an axiomatic theory of ascending truth and descending truth (ADT) and provide a possible-worlds semantics for it (which I dub *xeno semantics*). Xeno semantics is a generalization of the familiar neighborhood semantics, which itself is a generalization of the standard relational semantics. Once the details of ADT have been presented, it is easy to show that neither relational semantics nor neighborhood semantics will work for it; thus, the move to a more general framework is required. The main result is a fixed point theorem that guarantees the existence of an acceptable first-order constant-domain xeno model. From this result it follows that ADT is sound with respect to the class of such models. The upshot is that ADT is consistent relative to the background set theory.

1. A Theory of Ascending Truth and Descending Truth: ADT

Let $\mathcal{L}^{-}$ be a first-order language with the usual syntax (individual constants, variables, predicate letters, function letters, sentential operators, and quantifiers), the usual logical connectives ($\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$), existential and

[1] [4] and [5].

universal quantifers ($\forall$, $\exists$), and the vocabulary of Peano Arithmetic (0, successor function, addition and multiplication functions). Let L extend $\mathcal{L}$— by adding three one-place predicates, 'A(x)', 'D(x)', and 'S(x)' to be interpreted as an ascending truth predicate, a descending truth predicate, and a safety predicate, respectively. In what follows, I use lowercase Greek letters for variables ranging over well-formed formulas of $\mathcal{L}$ and angle brackets for names in $\mathcal{L}$ of well-formed formulas of $\mathcal{L}$; e.g., 'ϕ is a sentence of $\mathcal{L}$' and '$\langle\phi\rangle$ refers to ϕ'.

Consider the following list of axiom schemata.

D1 $D(\langle\phi\rangle) \rightarrow \phi$

D2 $D(\langle{\sim}\phi\rangle) \rightarrow {\sim}D(\langle\phi\rangle)$

D3 $D(\langle\phi\wedge\psi\rangle) \rightarrow D(\langle\phi\rangle) \wedge D(\langle\psi\rangle)$

D4 $D(\langle\phi\rangle) \vee D(\langle\psi\rangle) \rightarrow D(\langle\phi\vee\psi\rangle)$

D5 $D(\langle\phi\rangle)$ if ϕ is a tautology (logical truth) of first order predicate calculus.

D6 $D(\langle\phi\rangle)$ if ϕ is a theorem of PA.

D7 $D(\langle\phi\rangle)$ if ϕ is an axiom of ADT (i.e., if ϕ is an instance of D1-D6, A1-A6, M1-M4)

A1 $\phi \rightarrow A(\langle\phi\rangle)$

A2 ${\sim}A(\langle\phi\rangle) \rightarrow A(\langle{\sim}\phi\rangle)$

A3 $A(\langle\phi\rangle) \vee A(\langle\psi\rangle) \rightarrow A(\langle\phi\vee\psi\rangle)$

A4 $A(\langle\phi\wedge\psi\rangle) \rightarrow A(\langle\phi\rangle) \wedge A(\langle\psi\rangle)$

A5 ${\sim}A(\langle\phi\rangle)$ if ϕ is a contradiction (logical falsehood) of first order predicate calculus.

A6 ${\sim}A(\langle\phi\rangle)$ if ϕ is the negation of a theorem of PA.

M1 $D(\langle\phi\rangle) \leftrightarrow {\sim}A(\langle{\sim}\phi\rangle)$

M2 $S(\langle\phi\rangle) \leftrightarrow (D(\langle\phi\rangle) \vee {\sim}A(\langle\phi\rangle))$

M3 $\phi \wedge S(\langle\phi\rangle) \rightarrow D(\langle\phi\rangle)$

M4 $A(\langle\phi\rangle) \wedge S(\langle\phi\rangle) \rightarrow \phi$

E1 If $\sigma=\tau$ and ψ results from replacing some occurrences of σ with τ in ϕ, then $D(\langle\phi\rangle)\leftrightarrow D(\langle\psi\rangle)$.

E2 If $\sigma=\tau$ and ψ results from replacing some occurrences of σ with τ in ϕ, then $A(\langle\phi\rangle)\leftrightarrow A(\langle\psi\rangle)$.

E3 If $\sigma=\tau$ and ψ results from replacing some occurrences of σ with τ in ϕ, then $S(\langle\phi\rangle)\leftrightarrow S(\langle\psi\rangle)$.

Let ADT be the theory that consists of all the classical consequences of the instances of the above schemata. Note that there are redundancies in this list. There are reasons for including each of these principles, but elaborating them would take us too far from our current topic.

The ascending truth predicate obeys one direction (A1) of the familiar Tarskian schema-T, which is often taken to be constitutive of truth; the descending truth predicate obeys the other direction (D1). In addition, because of (M1), D(x) and A(x) are *dual* predicates—they have the same relationship that obtains between possibility and necessity, between obligation and permission, between provability and consistency, etc.[2]

2. Xeno Semantics

There are two major differences between the semantics I give for ADT and more familiar possible-worlds semantics. The first is that the most common form of possible worlds semantics, relational semantics, validates certain formulas and rules that are inconsistent with ADT; indeed, even the more general neighborhood semantics validates a rule that is inconsistent when paired with ADT. Therefore, neither of these semantics will work for ADT. The second difference is that ADT is a theory of three predicates, while modal logics are almost always theories of sentential operators, like the familiar necessity operator ($\Box$). I discuss these two differences in order.

In relational semantics, the extension of $\Box$ at each world is determined by a binary accessibility relation on the set of worlds; we can think of this as a function that assigns each world a set of worlds (i.e., those accessible from it). Moreover, each sentence is assigned a proposition, which is a set of worlds (i.e., those in which it is true). In neighborhood semantics, the extension of $\Box$ at each world is determined by a function that assigns each world a set of sets of worlds; as in relational semantics each sentence is assigned a proposition, which

[2] Dana Scott emphasized the importance of duality for ADT in private communication.

is a set of worlds. However, the problem with using relational semantics is that every relational model validates the so-called K axiom, $\Box(\phi\rightarrow\psi)\rightarrow(\Box\phi\rightarrow\Box\psi)$. Richard Montague proved that the predicate version of the K axiom (i.e., $\forall\phi\forall\psi(\Box(\langle\phi\rightarrow\psi\rangle)\rightarrow(\Box(\langle\phi\rangle)\rightarrow\Box(\langle\psi\rangle)))$ is inconsistent with the combination of D1, D5, D6, and D7 from ADT. Thus, no relational semantics can serve as a semantics for ADT.

Moreover, the problem with using neighborhood semantics is that every neighborhood model validates the so-called E-Rule:

(E) If $\vdash \phi\leftrightarrow\psi$, then $\vdash \Box\phi\leftrightarrow\Box\psi$.

However, the result of adding the predicate version of (E) to ADT is inconsistent. We know that when we make the move to first order logic and predicates, Godel's Diagonalization lemma guarantees that if our language can express Peano Arithmetic or its own theory of syntax (these are pretty minimal expressive constraints), then it will have sentences ϕ s.t. $\sim D(\langle\phi\rangle)$ is provably equivalent to ϕ. Let H be a sentence of $\mathcal{L}$ s.t. $H\leftrightarrow\sim D(\langle H\rangle)$. It is easy to show that ADT $\vdash \sim D(\langle H\rangle)$ (assume $D(\langle H\rangle)$; if $D(\langle H\rangle)$ then H; if H, then $\sim D(\langle H\rangle)$; so if $D(\langle H\rangle)$ then $\sim D(\langle H\rangle)$; thus, $\sim D(\langle H\rangle)$). Let J be any classical first-order tautology.[3] Since ADT $\vdash \sim D(\langle H\rangle)$, which is just H, ADT $\vdash H\leftrightarrow J$. By the predicate version of (E), ADT $\vdash D(\langle H\rangle)\leftrightarrow D(\langle J\rangle)$. However, ADT $\vdash D(\langle J\rangle)$. Thus, ADT $\vdash D(\langle H\rangle)$. $\perp$. This argument shows that the predicate version of rule E is incompatible with D1 and D5; similar arguments show that it is also incompatible with D1 and D6, and that it is incompatible with D1 and D7. These results show that no neighborhood semantics can serve as a semantics for ADT.[4]

In the new semantics, which I call *xeno semantics*[5], the extension of $\Box$ at each world is determined by both an accessibility relation *and* a neighborhood function. As before, each sentence is assigned a set of worlds as its proposition. However, although the neighborhood function is unchanged, the key to xeno semantics is that the accessibility relation is relative to each type of

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[3] Just to be clear, 'H' and 'J' are terms of the metalanguage, not L.

[4] Dana Scott first noticed this problem.

[5] Xeno semantics is named after our dog— thanks to Alison Duncan Kerr for the suggestion.

sentence in the language. Indeed, we can think of xeno semantics as involving as many binary accessibility relations as there are syntactic types of sentences.

In *relational* semantics $\Box\phi$ is true at a world w if and only if ϕ is true at all worlds accessible from w. In *neighborhood* semantics, $\Box\phi$ is true at a world w if and only if the set of worlds in which ϕ is true is a neighborhood of w. In *xeno* semantics, $\Box\phi$ is true at a world w if and only if the set of worlds in which ϕ is true is a neighborhood of all worlds accessible$_\phi$ from w, where 'accessible$_\phi$' is the accessibility relation assigned to ϕ's syntactic type. So one can think of xeno semantics as a blend of relational semantics and neighborhood semantics with a relativization to syntactic types. In xeno semantics, each sentence is assigned a proposition (a set of worlds) and a relation on the set of worlds. We can think of this as a sentence granting accessibility from one world to others, or we can say that the accessibility relation is relative to each sentence. Moreover, the accessibility relation alone does not determine the extension of $\Box$ at each world; rather, together the accessibility relation and the neighborhood relation determine the extension of $\Box$ *for that particular sentence* at each world. Alternatively, we can think of a proposition as a pair of a subset of W and a relation on W. But neighborhoods of a world are still just subsets of W. $\Box$'s extension at a world is then is an operation on propositions, and it is determined by the whole neighborhood function, not just the neighborhoods of that world. The details are given in the next section. The following diagram might illuminate the three kinds of semantics:

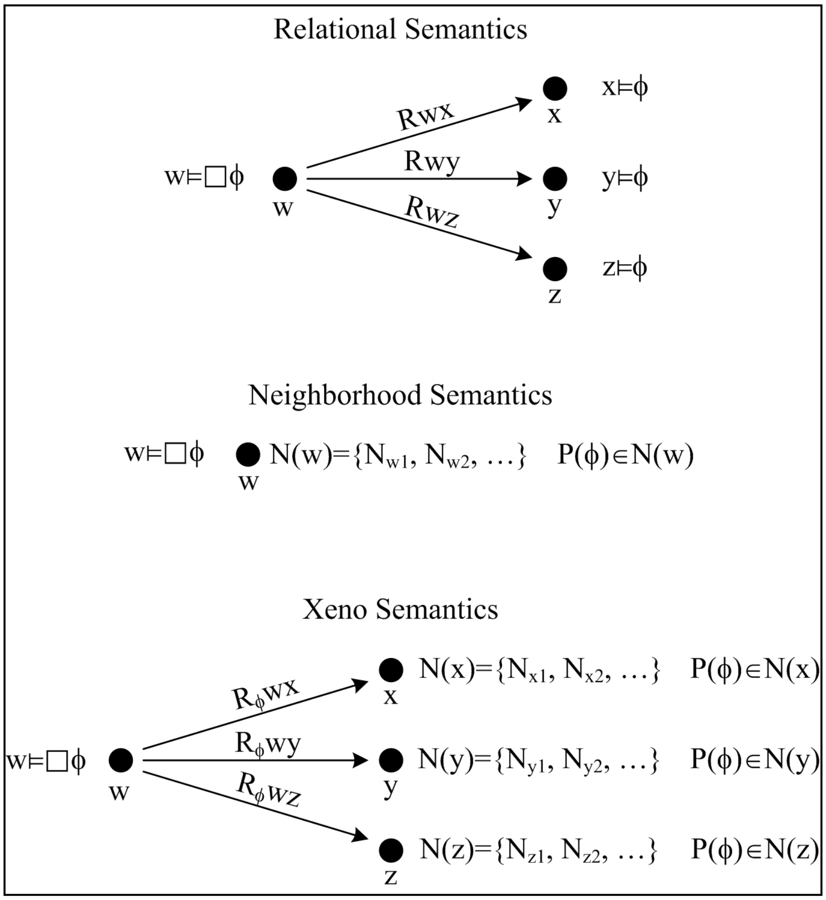

Figure 1

The second major difference between ADT and other theories for which possible-worlds semantics have been given (e.g., normal modal logics) is that ADT is a theory of three predicates, while possible-worlds semantics are almost always given for theories of sentential operators (e.g., a necessity operator). For example, let L_N be a sentential language with the usual syntax, the usual logical operators, and a sentential operator, $\Box$. Let W be a set of worlds and let R be a relation on W (called the *accessibility relation*). Together, W and R are called a *relational frame*, $\mathfrak{F}$=<W, R>. A valuation function, V, assigns to each sentential variable of L_N a truth value at each world in W. Together, $\mathfrak{F}$ and V are called a *relational model*, $\mathfrak{M}$=<$\mathfrak{F}$, V> for L_N. Each world in W is classical in that a classical scheme determines the value of truth-functionally compound sentences. That gives us the following clauses for defining truth at a world in a model (i.e., $<\mathfrak{M}, w>\models\phi$):

(ϕ) $<\mathfrak{M}, w>\models\phi$ if and only if $w\in V(\phi)$ for ϕ atomic.

($\sim$) $<\mathfrak{M}, w>\models\sim\phi$ if and only if it is not the case that $<\mathfrak{M}, w>\models\phi$

($\wedge$) $<\mathfrak{M}, w>\models\phi\wedge\psi$ if and only if $<\mathfrak{M}, w>\models\phi$ and $<\mathfrak{M}, w>\models\psi$

($\vee$) $<\mathfrak{M}, w>\models\phi\vee\psi$ if and only if $<\mathfrak{M}, w>\models\phi$ or $<\mathfrak{M}, w>\models\psi$

($\rightarrow$) $<\mathfrak{M}, w>\models\phi\rightarrow\psi$ if and only if if $<\mathfrak{M}, w>\models\phi$, then $<\mathfrak{M}, w>\models\psi$

($\leftrightarrow$) $<\mathfrak{M}, w>\models\phi\leftrightarrow\psi$ if and only if $<\mathfrak{M}, w>\models\phi$ iff $<\mathfrak{M}, w>\models\psi$

The clause for sentences of the form $\Box\phi$ is:

($\Box$) $<\mathfrak{M}, w>\models\Box\phi$ if and only if $\forall u\in W$ if Rwu, then $<\mathfrak{M}, u>\models\phi$

(i.e., $\Box\phi$ is true at w if and only if ϕ is true at all worlds accessible from w). One can use these clauses to provide an inductive definition of $<\mathfrak{M}, w>\models\phi$ based on the complexity of ϕ. A sentence ϕ is *valid in a model* $\mathfrak{M}$ (i.e., $\mathfrak{M}\models\phi$) if and only if $\forall w\in W$ $<\mathfrak{M}, w>\models\phi$. A sentence ϕ is *valid on a frame* $\mathfrak{F}$ (i.e., $\mathfrak{F}\models\phi$) if and only if for all $\mathfrak{M}$ based on $\mathfrak{F}$, $\forall w\in W$, $<\mathfrak{M}, w>\models\phi$.

However, in the case of ADT, we cannot adopt this kind of semantics since the inductive definition of $<\mathfrak{M}, w>\models\phi$ would not go through. The problem, of course, is that while $\Box\phi$ is syntactically more complex than ϕ since '$\Box$' is a sentential operator, $D(\langle\phi\rangle)$ need not be more complex than ϕ because '$D(x)$' is a predicate.

There has been some work done on using modal logic for predicates instead of operators, and one way to do it involves revision sequences. Revision sequences were originally designed to handle circular definitions, in which the definiens occurs as part of the definiendum. They can be adapted to modal logics for predicates by thinking of the definition of truth at a world in a model as a circular definition by virtue of the modal clauses. For example,

‘$D(\langle\phi\rangle)$’ can occur in the definiens for $<\mathfrak{M}, w>\vDash D(\langle\phi\rangle)$, which makes the overall definition circular. We can then use a revision sequence to arrive at particular frames and models.[6]

A revision sequence begins with an interpretation of the circularly defined term in question, and then one generates a sequence of interpretations through a revision rule, which is based on the circularly defined term. In our case, we start with a first order language that contains a predicate D(x), which will serve as our descending truth predicate (we will worry just about D(x) first, and then see if we can define A(x) and S(x) in terms of it). The revision sequence begins with a model of the language that is similar to the first-order xeno models discussed above, except this model will not satisfy the (D) clause. Instead, we use the (D) clause to generate a new model of the language, but it won’t satisfy the (D) clause either; by repeating this process over and over, we generate a sequence of models of the language. The goal is to reach a fixed point—i.e., a point in the sequence where it stops changing. If we can reach such a point, we would then have a legitimate definition of truth at a world in a xeno model for our descending truth *predicate*—a model of the language that satisfies the (D) clause.

3. A Fixed Point Theorem

Here we construct a revision sequence of xeno models and prove that it reaches a fixed point. Actually, our construction will be a bit more complicated—we first construct one revision sequence, Ω_0, using *neighborhood* semantics; we can think of this as our characterization sequence. It does not reach a fixed point, but it does classify our sentences in an illuminating way. We then use the results of this characterization sequence to construct the initial *xeno* model for a second revision sequence, Ω_1. The second revision sequence will eventually reach a fixed point. So we use a sequence of neighborhood models to construct a sequence of xeno models, and we prove that the sequence of xeno models reaches a fixed point. The fixed point for the sequence of xeno models will be a xeno model and it is our intended model for ADT.

3.1 The Characterization Sequence Ω_0

[6] Stewart Shapiro suggested this strategy to me. It is used in [1] and [2].

Again, $\mathcal{L}^-$ is a first order language with the usual connectives, quantifiers, individual constants, individual variables, and n-place predicates. We want $\mathcal{L}^-$ to have the resources to express its own syntax. The usual way of ensuring this is to stipulate that PA (Peano Arithmetic) is expressible in $\mathcal{L}^-$; however, there are some complications with this method that we explore below. We stipulate that PA is expressible in $\mathcal{L}^-$, but we also make sure that it's individual constants can directly refer to its own closed formulas by including all its closed formulas in the domain of any model for it.[7] Let $\mathcal{L}$ be the result of adding the predicate D(x) to $\mathcal{L}^-$. Let L^- be the set of well-formed formulas of $\mathcal{L}^-$ and let L be the set of well-formed formulas of $\mathcal{L}$.

We consider a neighborhood frame $\mathfrak{F}$=<W, N, $\mathfrak{D}$>, where W is a set of worlds, N is a neighborhood function from W to 2^{2^W}, and $\mathfrak{D}$ is the domain—a non-empty set. Let $\mathfrak{F}$ be an *suitable frame* if and only if:

(i) every neighborhood of every world in W is non-empty,

(ii) every world in W has a neighborhood,

We consider a neighborhood model $\mathfrak{M}$=<$\mathfrak{F}$, I>, where F is a suitable neighborhood frame, and I is an interpretation function.

Let $\mathfrak{M}_o$ = <W_0, N_0, $\mathfrak{D}_o$, I_0> be a neighborhood model based on a suitable frame, where:

(i) $\mathbb{N}\subset\mathfrak{D}_o$ (i.e., the domain contains the natural numbers),

(ii) $L\subset\mathfrak{D}_o$ (i.e., the domain contains the sentences of L),

(iii) $\forall w\in W_0$, I_0 assigns the arithmetic vocabulary in L to its standard interpretation in D_0,

(iv) $\forall\phi\in L\ \exists\sigma\ \sigma$ is an individual constant of $\mathcal{L}$ and $I_0(\sigma)=\phi$.

(v) $I_0(D(x), w)=\varnothing$ for all $w\in W_0$.

Let ν be a valuation (i.e., an assignment of elements from the domain to each individual variable of L).

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[7] The reason is that Gödel's diagonalization lemma guarantees the existence of sentences like H above that are provably equivalent to ~D(⟨H⟩). Note that 'H' need not refer to '~D(⟨H⟩)' for the two to be provably equivalent; nevertheless, provable equivalence is enough for most purposes. However, since the predicate version of rule E fails in ADT and xeno semantics, we need to require that 'H' actually refers to the formula '~D(⟨H⟩)' of $\mathcal{L}$. See [3] for discussion.

(F) $<\mathfrak{M}, w>\vDash_\nu F(a_1, \ldots, a_n)$ (where a_i is either an individual constant or an individual variable) if and only if $<f(a_1), \ldots, f(a_n)>\in I(F, w)$, where if a_i is a variable x_i, then $f(a_i)=\nu(x_i)$, and if a_i is an individual constant c_i, then $f(a_i)=I(c_i)$ (for each n-place predicate F).

($\sim$) $<\mathfrak{M}, w>\vDash_\nu\sim\phi$ if and only if it is not the case that $<\mathfrak{M}, w>\vDash_\nu\phi$

($\wedge$) $<\mathfrak{M}, w>\vDash_\nu\phi\wedge\psi$ if and only if $<\mathfrak{M}, w>\vDash_\nu\phi$ and $<\mathfrak{M}, w>\vDash_\nu\psi$

($\vee$) $<\mathfrak{M}, w>\vDash_\nu\phi\vee\psi$ if and only if $<\mathfrak{M}, w>\vDash_\nu\phi$ or $<\mathfrak{M}, w>\vDash_\nu\psi$

($\rightarrow$) $<\mathfrak{M}, w>\vDash_\nu\phi\rightarrow\psi$ if and only if if $<\mathfrak{M}, w>\vDash_\nu\phi$, then $<\mathfrak{M}, w>\vDash_\nu\psi$

($\leftrightarrow$) $<\mathfrak{M}, w>\vDash_\nu\phi\leftrightarrow\psi$ if and only if $<\mathfrak{M}, w>\vDash_\nu\phi$ iff $<\mathfrak{M}, w>\vDash_\nu\psi$

($\forall$) $<\mathfrak{M}, w>\vDash_\nu\forall x\phi(x)$ if and only if for each x-variant ν' $<\mathfrak{M}, w>\vDash_{\nu'}\phi(x)$

($\exists$) $<\mathfrak{M}, w>\vDash_\nu\exists x\phi(x)$ if and only if there is an x-variant ν' s.t. $<\mathfrak{M}, w>\vDash_{\nu'}\phi(x)$

We can say $<\mathfrak{M}, w>\vDash\phi$ if and only if ϕ is a closed formula and for all valuations ν, $<\mathfrak{M}, w>\vDash_\nu\phi$. Notice that the extension of the descending truth predicate, D(x), is stipulated to be empty in every world in M_0. Accordingly, $\mathfrak{M}_0$ has no clause for D(x).

$\mathfrak{M}_0$ will serve as the initial model for our first revision sequence. Before presenting the revision sequence, a few definitions are in order.

A ***revision rule*** ρ is an operation on the set of functions from $\{\{D(x)\} \times \mathfrak{D}\}$ to $\{t, f\}$. The members of this set of functions are ***hypotheses***. Each hypothesis interprets D(x). We focus on revision sequences Ω whose length, $lh(\Omega)$, is a limit ordinal or On, the class of all ordinals. Let $\Omega@\alpha$ be the αth member of Ω. Let $\Omega|\alpha$ be the restriction of Ω to ordinal α.

If $x\in\{t, f\}$ and $d\in\mathfrak{D}$, then d is ***stably x in*** Ω if and only if $\exists\beta$ s.t. $\beta<lh(\Omega)$ and for all ordinals γ, if $\beta\leq\gamma<ln(\Omega)$ then $[\Omega@\gamma](d) = x$; the least such β is the ***stabilization point*** of d in Ω. Say d is ***stable in*** Ω if and only if for some $x\in\{t, f\}$, d is stably x in Ω.

A hypothesis h ***coheres with a sequence*** Ω if and only if for all $d\in\mathfrak{D}$ and all $x\in\{t, f\}$, if d is stably x in Ω then $h(d)=x$.

Ω is a ***revision sequence*** for ρ if and only if for all $\alpha<lh(\Omega)$: (i) if $\alpha=\beta+1$, then $\Omega@\alpha = \rho(\Omega@\beta)$, and (ii) if α is a limit ordinal then $\Omega@\alpha$ coheres with $\Omega|\alpha$ (i.e., for all $d\in\mathfrak{D}$ and all $x\in\{t, f\}$ if d is stably x in $\Omega|\alpha$, then $\Omega@\alpha(d)=x$).[8]

These definitions are based on those in Gupta and Belnap (1993), which is the standard reference for revision sequences.

Our revision rule, which will generate the revision sequence, is based on the clause (D) that we would have wanted in our first order neighborhood semantics. Let Ω_0 be the revision sequence of length On with initial model $\mathfrak{M}_o$ generated by the following revision rule ρ_0:

(ρ_0-1) If α is not a limit ordinal, then $\forall w\in W$, if $\exists X\in N(w)$, s.t. $\forall x\in X$, $<\Omega_0@\alpha, x>\models\phi$, then $\phi\in I(D, w)$ for $\Omega_0@\alpha+1$; otherwise, $\phi\notin I(D, w)$ for $\Omega_0@\alpha+1$.

(ρ_0-2) If α is a limit ordinal and $D(\langle\phi\rangle)$ is stably true in $\Omega_0|\alpha$, then $\forall w\in W$, $\phi\in I(D, w)$ for $\Omega_0@\alpha$.

(ρ_0-3) If α is a limit ordinal and $D(\langle\phi\rangle)$ is stably false in $\Omega_0|\alpha$, then $\forall w\in W$, $\phi\notin I(D, w)$ for $\Omega_0@\alpha$.

(ρ_0-4) If α is a limit ordinal and $D(\langle\phi\rangle)$ is unstable in $\Omega_0|\alpha$, then $\forall w\in W$, $\phi\notin I(D, w)$ for $\Omega_0@\alpha$

The revision sequence based on this rule will have a fixed set of worlds and a fixed neighborhood function on that set. Obviously, the interpretation, I, changes from step to step, but the only difference between steps will be the interpretation of D(x). The interpretation of all other expressions in L does not change. One can think of this as a set of revision sequences, one for the

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[8] [1] chapter 5.

extension of D(x) at each world. Of course, at $\mathfrak{M}_0$, and indeed at each step throughout Ω_0, every world satisfies the same formulas.

3.2 The Primary Sequence Ω_1

Remember, Ω_0 is not the sequence we ultimately care about—its role is to help us assign accessibility relations to the sentences of L in a xeno semantics. That is, we use the results of Ω_0 to construct a new revision sequence of xeno models that *does* eventually reach a fixed point.

Not just any xeno frame and xeno model will do for these purposes. We need to define acceptable xeno frame and acceptable xeno model. There is one additional complication in the construction—we distinguish between traditional worlds (the set $C \subseteq W$) and non-traditional worlds (the set C'); the clause for D(x) is defined only on traditional worlds and validity is defined as truth at all traditional worlds. The extension of D(x) at non-traditional worlds is stipulated below.[9]

We will consider a constant domain xeno frame $\mathfrak{F} = \langle W, C, N, R, \mathfrak{D} \rangle$, where W is a set of worlds, $C \subseteq W$, N is a neighborhood function from W to 2^{2^W}, R is a denumerable set of binary relations on W, and $\mathfrak{D}$ is a non-empty set. Let $\mathfrak{F}$ be an *acceptable constant domain xeno frame* if and only if:

1. $C \subset W$ [non-traditional worlds]
2. $\forall w \in W, N(w) \neq \varnothing$ [all worlds have neighborhoods]
3. $\forall w \in W\ \forall X \in N(w), X \neq \varnothing$ [non-empty neighborhoods]
4. $\forall w \in W\ \forall X \in N(w), w \in X$ [inclusive neighborhoods]
5. $\forall v \in C'\ \forall X \in N(v)\ \forall x \in X, x \in C'$ [non-traditional neighborhoods]
6. $\forall u \in C, C \in N(u)$ [C is a traditional neighborhood]
7. $\forall w \in C, X \in N(w) \vee Y \in N(w) \rightarrow X \cup Y \in N(w)$ [supplemented neighborhoods]
8. If $X \subset C$ then $\forall u \in C, X \notin N(u)$ [no proper subset of C is a traditional neighborhood]

[9] There is a sense in which the logic determined by the particular xeno semantics I provide could be called a *non-traditional* modal logic in the spirit of non-normal modal logics and non-classical modal logics. I do not know if it is possible to avoid this aspect of the construction.

It should be obvious that acceptable constant domain xeno frames exist.

We consider a xeno model $\mathfrak{M}=<\mathfrak{F}, \mathfrak{R}, I>$ where $\mathfrak{F}$ is a xeno frame, $\mathfrak{R}$ is an accessibility function ($\mathfrak{R}$ is a function from L to R, so it assigns each sentence ϕ of L a binary relation on W, designated R_ϕ), and I is an interpretation function (I assigns each individual constant a member of the domain at each world and each n-place predicate a set of ordered n-tuples from the domain at each world). We use the following definitions for accessibility relations:

R_ϕ is *reflexive* if and only if $\forall w \in W\ R_\phi ww$

R_ϕ is *coreflexive* if and only if $\forall u \in W\ \forall w \in W,\ R_\phi wu \rightarrow w=u$

R_ϕ is *closed* if and only if $\forall u \in C\ \forall w \in W,\ R_\phi uw \rightarrow w \in C$

R_ϕ is *open* if and only if $\forall u \in C\ \exists v \in C',\ R_\phi uv$

Note that if an accessibility relation is coreflexive, then it is closed (but the converse fails). All accessibility relations in the xeno models we consider are reflexive. Intuitively, the accessibility relations assigned to the instances of axioms of ADT are coreflexive, as are those assigned to sentences of $\mathcal{L}^-$.

Let $\mathfrak{M}$ be an *acceptable xeno model* if and only if:

1. $\mathfrak{F}$ is an acceptable constant domain xeno frame
2. $\forall \phi \in L\ R_\phi$ is reflexive
3. If $I(\sigma)=I(\tau) \in L$, and ψ results from replacing occurrences of σ with τ in ϕ, then $R_\phi=R_\psi$.
4. $R_{\phi\wedge\psi}=R_{\psi\wedge\phi}$
5. $R_{\phi\vee\psi}=R_{\psi\vee\phi}$
6. $R_{\sim\sim\phi}=R_\phi$
7. R_ϕ is coreflexive for $\phi \in L^-$.
8. If R_ϕ is coreflexive then $R_{D\langle\phi\rangle}$ is coreflexive
9. $R_{D\langle\phi\rangle\rightarrow\phi}$ is coreflexive
10. $R_{D\langle\sim\phi\rangle \rightarrow \sim D\langle\phi\rangle}$ is coreflexive
11. $R_{D\langle\phi\wedge\psi\rangle\rightarrow D\langle\phi\rangle\wedge D\langle\psi\rangle}$ is coreflexive
12. $R_{D\langle\phi\rangle \vee D\langle\psi\rangle \rightarrow D\langle\phi\vee\psi\rangle}$ is coreflexive
13. R_ϕ is coreflexive for ϕ a first order classical logical truth.
14. R_ϕ is coreflexive for $PA \vdash \phi$
15. If R_ϕ is coreflexive and R_ψ is coreflexive then $R_{\phi\rightarrow\psi}$, $R_{\phi\wedge\psi}$, $R_{\phi\vee\psi}$ are coreflexive.

16. R_ϕ is coreflexive if and only if $R_{\sim\phi}$ is coreflexive.
17. If $R_{\phi\wedge\psi}$ is closed and $C\subseteq P_v(\phi\wedge\psi)$, then R_ϕ is closed and R_ψ is closed.[10]
18. If R_ϕ is closed and R_ψ is closed then $R_{\phi\vee\psi}$ is closed.
19. $L\subset\mathfrak{D}$ (i.e., the domain contains all closed sentences of L)
20. $\mathbb{N}\subset\mathfrak{D}$ (i.e., the domain contains the natural numbers)
21. $\forall w\in W$, I assigns the arithmetic vocabulary in $\mathcal{L}$ to their standard interpretation in $\mathfrak{D}$
22. All individual constants have the same denotation in every world.
23. All predicates (except possibly D) have the same extension in every world.
24. All predicates have the same extension in all non-traditional worlds.
25. $\forall\phi\in L$, $\exists\sigma$ σ is an individual constant of $\mathcal{L}$ and $I(\sigma)=\phi$
26. If R_ϕ is coreflexive then $C\subseteq P_v(\phi)$ or $C\subseteq P_v(\sim\phi)$

Let Δ_M be the set of sentences of $\mathcal{L}$ closed under the following rules:

($\Delta_{\mathfrak{M}}$ -1) $\forall\phi\in L$ if $\phi=D\langle\psi\rangle\rightarrow\psi$ then $\phi\in\Delta_{\mathfrak{M}}$

($\Delta_{\mathfrak{M}}$ -2) $\forall\phi\in L$ if $\phi\in\Delta_{\mathfrak{M}}$ and σ is an individual constant and $I(\sigma)=\phi$, then $D\sigma\in\Delta_{\mathfrak{M}}$

That is, $\Delta_{\mathfrak{M}}$ is the set of instances of axiom schema D1 closed under applications of D.

Let $\mathfrak{M}_1$ = $<W_1, C_1, N_1, R_1, \mathfrak{D}_1, \mathfrak{R}_1, I_1>$ be an acceptable constant domain xeno model, where:

(i) $I_1(D(x), w)=\varnothing$ for all $w\in W_1$.

(ii) if ϕ is stably true in Ω_0, then $R_{1\phi}$ is closed in $\mathfrak{M}_1$.

(iii) if ϕ is stably false in Ω_0, then $R_{1\phi}$ is closed in $\mathfrak{M}_1$

(iv) if ϕ is unstable in Ω_0 and $\phi\notin\Delta_{M1}$ then $R_{1\phi}$ is open in $\mathfrak{M}_1$.

There are several issues to be settled before we can be sure that $\mathfrak{M}_1$ exists.

First, we need to show that there are acceptable constant domain xeno models. That is easy—let $\mathfrak{M}$ be a constant domain xeno model based on an acceptable xeno frame such that the natural numbers and the closed formulas of $\mathcal{L}$ are members of its domain, the arithmetic vocabulary of $\mathcal{L}$ receives its standard interpretation in every world, there is a name in $\mathcal{L}$ for every member

[10] For each ϕ in L, $P_v(\phi)=\{w\in W: <\mathfrak{M}, w>\vDash_v \phi\}$

of the domain, the interpretation is the same at every world, and every relation in R is the identity relation. Then $\mathfrak{M}$ is an acceptable constant domain xeno model.

Second, we need to show that $\mathfrak{R}_1$ is well-defined. Given the interpretation function I_1, we define $\Delta_{\mathfrak{M}_1}$ as above. There are eighteen conditions under the definition of an acceptable xeno model that pertain to $\mathfrak{R}_1$. None of them conflict with the above specification that defines $\mathfrak{M}_1$. For example, condition 4 is: $R_{1_{\phi\wedge\psi}}=R_{1_{\psi\wedge\phi}}$. It is obvious that $\phi\wedge\psi$ is stable in Ω_0 if and only if $\psi\wedge\phi$ is stable in Ω_0; thus the specification of R_1 does not conflict with this condition. The same holds for all the others. One might worry about condition 9, but all instances of D1 are in $\Delta_{\mathfrak{M}_1}$, so that does not pose a problem. Thus, the above specification of the accessibility relations in M_1 does not conflict with the definition of an acceptable xeno model. Therefore, $\mathfrak{R}_1$ is well-defined.

Let ν be a valuation (i.e., n assignment of elements from the domain to each individual variable of $\mathcal{L}$).

(F) $<\mathfrak{M}, w>\vDash_\nu F(a_1, \ldots, a_n)$ (where a_i is either an individual constant or an individual variable) if and only if $<f(a_1), \ldots, f(a_n)>\in I(F, w)$, where if a_i is a variable x_i, then $f(a_i)=\nu(x_i)$, and if a_i is an individual constant c_i, then $f(a_i)=I(c_i)$ (for each n-place predicate F).

(~) $<\mathfrak{M}, w>\vDash_\nu\sim\phi$ if and only if it is not the case that $<\mathfrak{M}, w>\vDash_\nu\phi$

(∧) $<\mathfrak{M}, w>\vDash_\nu\phi\wedge\psi$ if and only if $<\mathfrak{M}, w>\vDash_\nu\phi$ and $<\mathfrak{M}, w>\vDash_\nu\psi$

(∨) $<\mathfrak{M}, w>\vDash_\nu\phi\vee\psi$ if and only if $<\mathfrak{M}, w>\vDash_\nu\phi$ or $<\mathfrak{M}, w>\vDash_\nu\psi$

(→) $<\mathfrak{M}, w>\vDash_\nu\phi\rightarrow\psi$ if and only if if $<\mathfrak{M}, w>\vDash_\nu\phi$, then $<\mathfrak{M}, w>\vDash_\nu\psi$

(↔) $<\mathfrak{M}, w>\vDash_\nu\phi\leftrightarrow\psi$ if and only if $<\mathfrak{M}, w>\vDash_\nu\phi$ iff $<\mathfrak{M}, w>\vDash_\nu\psi$

(∀) $<\mathfrak{M}, w>\vDash_\nu\forall x\phi(x)$ if and only if for each x-variant ν' $<\mathfrak{M}, w>\vDash_{\nu'}\phi(x)$

(∃) $<\mathfrak{M}, w>\vDash_\nu\exists x\phi(x)$ if and only if there is an x-variant ν' s.t. $<\mathfrak{M}, w>\vDash_{\nu'}\phi(x)$

We can say $<\mathfrak{M}, w>\vDash\phi$ if and only if ϕ is a closed formula and for all valuations ν, $<\mathfrak{M}, w>\vDash_\nu\phi$.

Notice that the extension of the descending truth predicate, D(x), is empty in every world in $\mathfrak{M}_1$. Accordingly, $\mathfrak{M}_1$ has no clause for D(x). Let Ω_1 be the revision sequence of length On with initial model $\mathfrak{M}_1$ generated by the following revision rule ρ_1:

(ρ_1-1) α is not a limit ordinal: if $\forall u \in C$, $\forall w \in W$, $R_\phi uw \rightarrow P_{\alpha-1}(\phi) \in N(w)$, then $\forall w \in W$, $\phi \in I(D, w)$ for $\Omega_1@\alpha$; otherwise, $\forall w \in W$, $\phi \notin I(D, w)$ for $\Omega_1@\alpha$.

(ρ_1-2) α is a limit ordinal: if $D\langle\phi\rangle$ is stably true in $\Omega_1|\alpha$, then $\forall w \in C$, $\phi \in I(D, w)$ for $\Omega_1@\alpha+1$.

(ρ_1-3) α is a limit ordinal: if $D\langle\phi\rangle$ is stably false in $\Omega_1|\alpha$, then $\forall w \in C$, $\phi \notin I(D, w)$ for $\Omega_1@\alpha+1$.

(ρ_1-4) α is a limit ordinal: if $D\langle\phi\rangle$ is unstable in $\Omega_1|\alpha$, then $\forall w \in C$, $\phi \notin I(D, w)$ for $\Omega_1@\alpha+1$.

The revision sequence based on this rule will have a fixed set of worlds and a fixed neighborhood function on that set. As before, the interpretation, I, changes from step to step, but the only difference between steps will be the interpretation of D(x). The interpretation of all other expressions in $\mathcal{L}$ does not change. The assignment of accessibility relations to sentences of $\mathcal{L}$ does not change.

3.3 A Fixed Point for Ω_1

Now we prove that Ω_1 reaches a fixed point. Before we do that, we need several more definitions and results pertaining to revision sequences (I omit the proofs).

A hypothesis h is ***cofinal in a sequence*** Ω if and only if for all ordinals $\alpha<\ln(\Omega)$ there is a β s.t. $\alpha\leq\beta<\ln(\Omega)$ and $\Omega@\beta=h$.

Theorem: Ω is a sequence of length On. Then:

(i) there is a hypothesis $h\in\{t, f\}^{\mathfrak{D}}$ that is cofinal in Ω

(ii) there is an ordinal α s.t. for all $\beta\geq\alpha$, $\Omega@\beta$ is confinal in Ω; the least such ordinal is the ***initial ordinal*** for Ω.

(iii) for all ordinals α there is an ordinal $\beta>\alpha$ satisfying the condition that for all hypotheses h cofinal in Ω there is an ordinal γ s.t. $\alpha\leq\gamma<\beta$ and $\Omega@\gamma=h$; such an ordinal is a ***completion ordinal*** for Ω above α.

Theorem: for all $d\in\mathfrak{D}$ and $x\in\{t, f\}$,

(i) if d is stably x in Ω then the value of d is x in all hypotheses cofinal in Ω

(ii) if $lh(\Omega)=On$, then the converse of (i) is true.

An ordinal α is a ***reflection ordinal*** for Ω if and only if α is a limit ordinal $< lh(\Omega)$ s.t.

(i) $\alpha \geq$ the initial ordinal for Ω, and

(ii) for all $d\in\mathfrak{D}$ and $x\in\{t, f\}$, d is stably x in $\Omega|\alpha$ if and only if d is stably x in Ω.

Theorem: Let Ω be a revision sequence for ρ and $\alpha<lh(\Omega)$. If $\Omega@\alpha$ is a fixed point of ρ then for all β s.t. $\alpha+\beta < lh(\Omega)$ we have $\Omega@\alpha+\beta = \Omega@\alpha$; furthermore, an object $d\in\mathfrak{D}$ is stably x in Ω if and only if $\Omega@\alpha(d)=x$.

A hypothesis h is ***recurring*** for ρ if and only if h is cofinal in some revision sequence Ω of length On for ρ.

Theorem: all and only recurring hypotheses are reflexive.[11]

So if α is a reflection ordinal, then $\Omega|\alpha$ reflects all the stabilities and instabilities in Ω.

With these definitions and results in hand, we are ready to show that Ω_1 reaches a fixed point. I use the convention '$P_\alpha(\phi)$' for $\{w\in W : <\Omega_1@\alpha, w>\vDash\phi\}$.

[11] For proofs, see [1] chapter 5.

Let ζ be the initial ordinal for Ω_1, and let ξ be a reflection ordinal for Ω_1 s.t. $\xi>\zeta$, and let $\mathfrak{M}_2=\Omega_1@\xi$. Thus, $\mathfrak{M}_2$ is a reflexive hypothesis for Ω_1; it follows that ξ is a limit ordinal.

I rely on the following lemmas:

(1) ϕ is stable in $\Omega_1|\xi$ if and only if ϕ is stable in Ω_1. [ξ is a reflection ordinal for Ω, so by definition, ϕ is stable in $\Omega|\xi$ if and only if ϕ is stable in Ω.]

(2) For any ordinal α, if $\exists w\in C$ s.t. $<\Omega_1@\alpha, w>\vDash\phi$, then $\forall w\in C$, $<\Omega_1@\alpha, w>\vDash\phi$. [By induction—the extension of D(x) is the same at all classical worlds in $\mathfrak{M}_1$, and if the extension of D(x) is the same at all classical worlds in $\Omega@\alpha$, then the extension of D(x) is the same at all classical worlds in $\Omega@\alpha+1$.]

To show that $\mathfrak{M}_2$ is a fixed point for ρ_1, we prove that the extension of D(x) does not change from $\mathfrak{M}_2$ to $\rho_1(\mathfrak{M}_2)$ on traditional worlds.

Let $u\in C$. Let $Q\in L$. Assume that $<\mathfrak{M}_2, u>\vDash D\langle Q\rangle$. Assume for reductio that $<\rho_1(\mathfrak{M}_2), u>\vDash{\sim}D\langle Q\rangle$. $D\langle Q\rangle$ is stably true at u in $\Omega_1|\xi$ [else $<\mathfrak{M}_2$, $u>\vDash{\sim}D\langle Q\rangle$, since ξ is a limit ordinal]. $D\langle Q\rangle$ is stably true at u in Ω_1 [by Lemma 1]. $<\rho_1(\mathfrak{M}_2), u>\vDash D\langle Q\rangle$. $\bot$. This result shows that the extension of D(x) does not decrease from $\mathfrak{M}_2$ to $\rho_1(\mathfrak{M}_2)$. Now for the other direction.

Assume that $<\mathfrak{M}_2, u>\vDash{\sim}D\langle Q\rangle$. Assume for reductio 1 that $<\rho_1(\mathfrak{M}_2)$, $u>\vDash D\langle Q\rangle$. $\forall w\in C$, $R_Quw \rightarrow P_\xi(Q)\in N(w)$. It follows that R_Quu. Hence, $P_\xi(Q)\in N(u)$. Thus, $\forall X\in N(u)$, $u\in X$. Therefore, $u\in P_\xi(Q)$, and it follows that $<\mathfrak{M}_2, u>\vDash Q$. Either $D\langle Q\rangle$ is stably false at u in $\Omega_1|\xi$ or $D\langle Q\rangle$ is unstable at u in $\Omega_1|\xi$ [else $<\mathfrak{M}_2, u>\vDash D\langle Q\rangle$]. Assume for reductio 2 that $D\langle Q\rangle$ is stably false at u in $\Omega_1|\xi$. It follows that $D\langle Q\rangle$ is stably false at u in Ω_1 [by Lemma 1]. Hence, $<\rho_1(\mathfrak{M}_2), w>\vDash{\sim}D\langle Q\rangle$. $\bot$ (for reductio 2). Now for the other disjunct. Assume for reductio 3 that $D\langle Q\rangle$ is unstable at u in $\Omega_1|\xi$. $D\langle Q\rangle$ is unstable at u in Ω_1 [by Lemma 1]. Assume for conditional proof that Q is stable at u in Ω_1. Hence, $\forall w\in C$, Q is stable at w in Ω_1 [by Lemma 2]. Let β be the stabilization point for Q at u in Ω_1. Then, $\forall\gamma>\beta\ u\in P_\gamma(Q)$ or $\forall\gamma>\beta\ u\notin P_\gamma(Q)$. Hence, $\forall w\in C\ (\forall\gamma>\beta\ w\in P_\gamma(Q)$ or $\forall\gamma>\beta\ w\notin P_\gamma(Q))$. Thus, $\forall w\in W\ (\forall\gamma>\beta\ w\in P_\gamma(Q)$ or $\forall\gamma>\beta\ w\notin P_\gamma(Q))$. Hence, $\forall\gamma>\beta\ P_\gamma(Q)\in N(u)$ or $\forall\gamma>\beta\ P_\gamma(Q)\notin N(u)$. R_Q is either open or closed; if R_Q is open, then $\sim D\langle Q\rangle$ is stably false at u in Ω_1. Thus, R_Q is closed. Either $\forall\gamma>\beta\ <\Omega_1@\gamma, u>\vDash D\langle Q\rangle$ or $\forall\gamma>\beta\ <\Omega_1@\gamma, u>\vDash\sim D\langle Q\rangle$. Hence, $D\langle Q\rangle$ is stable at u in Ω_1. By conditional proof, if Q is stable at u in Ω_1 then $D\langle Q\rangle$ is stable at u in Ω_1. So, by contraposition, if $D\langle Q\rangle$ is unstable at u in Ω_1 then Q is unstable at u in Ω_1. Thus, Q is unstable at u in Ω_1. Hence, Q is unstable at u in $\Omega_1|\xi$. Therefore, $<\mathfrak{M}_2, u>\vDash\sim Q$. We have $\perp$ for reductio 3. And we have $\perp$ for reductio 1. Consequently, we have a fixed point, and $\mathfrak{M}_2$ is the intended model for $\mathcal{L}$.

Since $\mathfrak{M}_2$ is a fixed point for ρ_1, we know that for $u\in C$:

(D) $<\mathfrak{M}_2, u>\vDash D(\langle\phi\rangle)$ if and only if $\forall w\in W\ R_\phi uw \rightarrow P(\phi)\in N(w)$

So we have a constant domain xeno semantics for $\mathcal{L}$, and it satisfies the intended clause for D(x). We have not said anything about the non-traditional worlds (we have not needed to say anything about them), but to finish the interpretation of $\mathcal{L}$, we can say that if for all $w\in C\ w\vDash\phi$, then for all $v\in C'\ v\vDash\phi$.

Recall that we have been concentrating on D(x) and ignoring A(x) and S(x). If we can define them in terms of D(x), then that would do the trick. We could use the following definitions:

(M1) $A(\langle\phi\rangle) \leftrightarrow \sim D(\langle\sim\phi\rangle)$

(M2) $S(\langle\phi\rangle) \leftrightarrow D(\langle\phi\rangle) \vee \sim A(\langle\phi\rangle)$

Let $\mathcal{L}^+$ be the result of adding A(x) and S(x) to $\mathcal{L}$ and let L^+ be the set of formulas of $\mathcal{L}^+$. Here is how to interpret the new predicates. Let $\phi\in L^+/L$. Let ψ result from replacing all occurrences of $A\langle\theta\rangle$ in ϕ with $\sim D\langle\sim\theta\rangle$ and replacing all occurrences of $S\langle\theta\rangle$ in ϕ with $D\langle\theta\rangle\vee D\langle\sim\theta\rangle$. Then $R_\phi=R_\psi$ and $\forall w\in W\ w\vDash\phi$ if and only if $w\vDash\psi$.

To summarize the constant domain xeno semantics for the descending truth predicate, 'D(x)', the ascending truth predicate, 'A(x)', and the safety predicate, 'S(x)':

(F) $<\mathfrak{M}, w>\vDash_v F(a_1, \ldots, a_n)$ (where a_i is either an individual constant or an individual variable) if and only if $<f(a_1), \ldots, f(a_n)>\in I(F, w)$, where if a_i is a variable x_i, then $f(a_i)=\nu(x_i)$, and if a_i is an individual constant c_i, then $f(a_i)=I(c_i)$ (for each n-place predicate F).

(~) $<\mathfrak{M}, w> \vDash_v \sim\phi$ if and only if it is not the case that $<\mathfrak{M}, w> \vDash_v \phi$

(∧) $<\mathfrak{M}, w> \vDash_v \phi \wedge \psi$ if and only if $<\mathfrak{M}, w> \vDash_v \phi$ and $<\mathfrak{M}, w> \vDash_v \psi$

(∨) $<\mathfrak{M}, w> \vDash_v \phi \vee \psi$ if and only if $<\mathfrak{M}, w> \vDash_v \phi$ or $<\mathfrak{M}, w> \vDash_v \psi$

(→) $<\mathfrak{M}, w> \vDash_v \phi \rightarrow \psi$ if and only if if $<\mathfrak{M}, w> \vDash_v \phi$, then $<\mathfrak{M}, w> \vDash_v \psi$

(↔) $<\mathfrak{M}, w> \vDash_v \phi \leftrightarrow \psi$ if and only if $<\mathfrak{M}, w> \vDash_v \phi$ iff $<\mathfrak{M}, w> \vDash_v \psi$

(∀) $<\mathfrak{M}, w> \vDash_v \forall x \phi(x)$ if and only if for each x-variant ν' $<\mathfrak{M}, w> \vDash_{v'} \phi(x)$

(∃) $<\mathfrak{M}, w> \vDash_v \exists x \phi(x)$ if and only if there is an x-variant ν' s.t. $<\mathfrak{M}, w> \vDash_{v'} \phi(x)$

For all $u \in C$:

(D) $<\mathfrak{M}, u> \vDash_v D\langle\phi\rangle$ if and only if $\forall w \in W\ R_\phi uw \rightarrow P_v(\phi) \in N(w)$

(A) $<\mathfrak{M}, u> \vDash_v A\langle\phi\rangle$ if and only if $\exists w \in W\ R_{\sim\phi} uw \wedge P_v(\sim\phi) \notin N(w)$

(S) $<\mathfrak{M}, u> \vDash_v S\langle\phi\rangle$ if and only if $\forall w \in W\ (R_\phi uw \rightarrow P_v(\phi) \in N(w)) \vee \exists u \in W\ (R_{\sim\phi} uw \wedge P_v(\sim\phi) \notin N(w))$

For all $v \in C'$:

(D) $<\mathfrak{M}, v> \vDash_v D\langle\phi\rangle$ if and only if $\forall u \in C$ $<\mathfrak{M}, u> \vDash_v D\langle\phi\rangle$

(A) $<\mathfrak{M}, v> \vDash_v A\langle\phi\rangle$ if and only if $\forall u \in C$ $<\mathfrak{M}, u> \vDash_v A\langle\phi\rangle$

(S) $<\mathfrak{M}, v> \vDash_v S\langle\phi\rangle$ if and only if $\forall u \in C$ $<\mathfrak{M}, u> \vDash_v S\langle\phi\rangle$

We can say $<\mathfrak{M}, w> \vDash \phi$ if and only if ϕ is a closed formula and for all valuations ν, $<\mathfrak{M}, w> \vDash_v \phi$.

A sentence ϕ is *valid* in a xeno model $\mathfrak{M}$ if and only if $\forall u \in C$ $<\mathfrak{M}, u> \vDash \phi$

4. Soundness

The fixed point theorem from the previous section shows that we have a well-defined notion of truth at a world for an acceptable constant domain xeno model. It follows that we have a well-defined notion of validity for constant domain xeno models. Now, all that is left is to show that ADT is sound with respect to the class of acceptable constant domain xeno models. It follows from this result that ADT is consistent (relative to our background set theory).

In order to prove soundness, we need to go through each of the axioms of ADT and prove that they are valid in any acceptable xeno model. It is a tedious but trivial exercise to demonstrate this, and I do not give the details here.[12] The result is, if ϕ is an axiom of ADT, then ϕ is valid in any acceptable constant domain xeno model.

ADT is sound with respect to constant domain xeno semantics if and only if for all acceptable constant domain xeno models μ, any set of sentences Γ and any sentence ϕ, if ϕ is provable from Γ, then the argument from Γ to ϕ is valid. Argument validity is defined in the usual way: for all acceptable constant domain xeno models μ if all the members of Γ are true in μ, then ϕ is true in μ. We know that all the classical logical truths are valid and all classical inference rules are valid. So our proof that all axioms of ADT are valid in any acceptable constant domain xeno model completes our soundness proof. ADT is sound with respect to xeno semantics.

[12] For example, to show that all instances of axiom schema D1 are valid, assume for an acceptable constant domain xeno model that $u \in C$ $u \vDash D\langle\phi\rangle$. It follows that $\forall w \in W$ $R_\phi uw \rightarrow P(\phi) \in N(w)$. By clause 2 of the definition of an acceptable xeno model, R_ϕ is reflexive. Thus, Ruu. Hence, $P(\phi) \in N(u)$. By clause 4 of the definition of an acceptable xeno frame, N is inclusive. Thus, $u \in P(\phi)$. Hence, $u \vDash \phi$. Therefore, $\vDash D\langle\phi\rangle \rightarrow \phi$. The proofs for the other axioms are similar.

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